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# Eulerian Model for Mean Turbulent Diffusion of Particles in Free Shear Layers

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**A simple Eulerian model is developed to determine the qualitative mean transverse diffusion in a free shear flow of a particle with a drag coefficient inversely proportional to the particle Reynolds number. The model is based on integrating the particle equation of motion within a single time-varying eddy to yield a closed-form analytic expression for the turbulent diffusion. The description is based on an average local length scale and an average local timescale of the turbulence similar to previous Lagrangian stochastic diffusion models. The resulting expression indicates three important nondimensional parameters: a local Stokes number (ratio of particle response time to eddy lifetime), an eddy Froude number (ratio of rotational acceleration to gravitational acceleration), and a drift parameter (ratio of particle terminal velocity to turbulent rms fluctuations), where the third parameter is simply a function of the first two. The model yields a particle diffusion greater than that of a scalar for a local Stokes number of order unity, the value of which depends on the eddy Froude number. Comparison with available experimental data shows qualitative agreement despite the simplicity and nonempiricism of the model.**

## Nomenclature

$b$ = local spatial spread rate
$C_D$ = drag coefficient
$c_1$ = velocity scale constant of the continuous phase
$c_2$ = length scale constant of the continuous phase
$\bar{D}$ = diffusion ratio
$\boldsymbol{D}$ = drag force on particle
$d$ = diameter or equivalent volumetric diameter
$Fr_\delta$ = eddy Froude number, $\Delta u^2/4g\delta$
$\boldsymbol{G}$ = gravitational force on particle
$g$ = acceleration due to gravity
$k_1$ = drag coefficient constant
$m_p$ = particle mass
$Re_p$ = particle Reynolds number based on $V_f - V_p$
$S$ = local Stokes number, $\tau_p/\tau_e$
$u$ = streamwise velocity
$\boldsymbol{V}$ = velocity vector
$V_{\text{rel}}$ = relative velocity of the particle to the continuous fluid
$V_{\text{term}}$ = terminal velocity of the particle
$v$ = transverse velocity
$x$ = streamwise position
$y$ = transverse position
$\beta$ = ratio of interaction timescale to eddy lifetime
$\gamma$ = drift parameter, $V_{\text{term}}/c_1\Delta u$
$\Delta u$ = velocity difference between the high-speed stream and the low-speed stream
$\delta$ = shear layer thickness
$\varepsilon$ = $+1$ or $-1$ depending on whether initial fluctuation velocity is positive or negative
$\theta$ = component of gravity in a given direction
$\lambda$ = eddy integral scale
$\mu$ = viscosity
$\rho$ = density
$\tau_e$ = eddy lifetime
$\tau_i$ = eddy interaction timescale
$\tau_p$ = particle response time
$\tau_{\text{transit}}$ = particle traverse time

*Subscripts*

$f$ = continuous fluid
$m$ = mean convection
$p$ = particle
$x$ = streamwise direction
$y$ = transverse direction
1 = high-speed stream value
2 = low-speed stream value

## Introduction

TURBULENT dispersion of particles and droplets in free shear flows is important in many two-phase flow areas, such as sprays and particle combustion. Two important aspects that need to be understood to model such flows are characterization of the detailed spatiotemporal structures of particle concentration due to the underlying turbulence of the continuous phase[1] and the characterization of the mean diffusion rates of the dispersed phase, which is the subject of the present work. The mean diffusion has been found experimentally to vary considerably as a function of the Stokes number (ratio of particle response time to eddy timescale) in both jets and planar shear layers. In particular, very small particles diffuse at a rate similar to that of a scalar field and very large particles diffuse at much slower rates than that of a scalar field inasmuch as their inertia is too great to be significantly affected by the turbulent dispersion; however, particles with Stokes numbers of order unity can have mean diffusion rates in excess of that for a scalar.[2] Therefore, the diffusion ratio $\bar{D}$, defined as the ratio of particle diffusion to scalar diffusion, is generally found to approach unity for Stokes numbers much less than unity, to exceed unity for Stokes numbers of order unity, and to approach zero for Stokes numbers much greater than unity. This nonmonotonic behavior is a result of intermediate particles (of order unity Stokes number) being dispersed outward by a velocity pertubation but then not responding quick enough to return with the fluid with a subsequent opposing velocity perturbation.

The nonmonotonic variation of particle diffusion rates with respect to Stokes number has also been observed numerically in several Lagrangian studies of particle transport that incorporate unsteady eddy-resolved dynamics of the continuous phase. Such studies range from two-dimensional discrete-vortex simulations[2] to three-dimensional direct numerical simulations[3] with an aim to resolve the unsteady turbulent structures directly such that no empirical model to correlate particle dispersion with mean turbulence intensity is neccesary. However, such eddy-resolved numerical approaches are impractical with respect to computational resources for the majority of complex-geometry engineering calculations of turbulent flows. In such cases, Reynolds averaged Navier–Stokes (RANS) computations solved in an Eulerian frame are typically used for the continuous phase, where the time-averaged velocities and turbulence intensities of the continuous phase are obtained by using

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empirical closure arguments for the turbulence correlations. These RANS computations are conventionally divided into Lagrangian and Eulerian treatments of the dispersed phase.

The Lagrangian models conventionally include stochastic eddy interaction models with a statistically large number of particles to predict turbulent diffusion. For example, the Lagrangian approaches classified as discontinuous random walk (DRW) models employ a stochastic velocity perturbation that is held constant throughout an eddy interaction cycle where the interaction time is limited to the minimum of the traverse time and the eddy timescale. These simple models have shown success in predicting dispersion in turbulent wakes and jets,[4] although more advanced stochastic Lagrangian models allow for time-varying velocity perturbations and can eliminate spurious drift velocities[5] to further improve the modeling of the particle turbulent diffusion. However, achieving statistical convergence with a Lagrangian approach typically requires more computational resources than are used for an Eulerian treatment of the dispersed phase. In addition, an Eulerian approach allows both phases to be handled with a consistent numerical scheme and a consistent numerical grid. The discretization coincidence for Eulerian treatment of the particles becomes a distinct accuracy advantage when one is trying to compute the effects of the particles on the continuous fluid.[6]

The more computationally convenient Eulerian method typically assumes that both the carrier fluid and the particles comprise two separate, but intermixed, continua. The equation for the mass diffusion of the particle concentration typically employs a gradient diffusion model yielding a time-averaged continuity equation that requires a local model for the mean particle diffusion, i.e., an equation for $\bar{D}$. Even the more complex large eddy simulation (LES) approach yields a subgrid scale for which the subgrid Eulerian turbulent diffusivity must be estimated. However, current Eulerian models for mean turbulent diffusion of particles do not accurately represent recent experimental measurements, e.g., cannot predict the nonmonotonic behavior and/or even robustly predict the reduced diffusion associated with large particles.[7] One reason is that many of these models do not formally include the particle equation of motion and the crossing-trajectory effect. As a result, the models are limited to dependence on Stokes number only where proper treatment requires an additional dependence on an eddy Froude number[8] or a drift parameter.[9] There are a few notable exceptions. The analytical models of Reeks[8] and Stock[9] do yield a finite nonmonotonic peak of particle diffusion above the scalar diffusion for large particles, although this peak was noted only for eddy interaction times much greater than the eddy timescale (inconsistent with the aforementioned DRW interaction times). The model of Hunt et al.[10] yields nonmonotonic behavior for Stokes numbers of order unity by employing two length scales, although it allows for only an infinitesimally small peak of particle diffusion above the scalar diffusion.

The objective of this study is to develop an Eulerian model of turbulent particle diffusion that reproduces the finite nonmonotonic effects noted in recent free-shear-flow experiments while employing interaction times and eddy characteristics consistent with the Lagrangian DRW models. The difference is that the present model employs a deterministic time-varying, single-eddy interaction. This allows comparison of the particle and scalar trajectories to evaluate a ratio between the mean diffusion rates. In doing so, we wish to determine the effect of the critical nondimensional parameters in describing the mean transverse particle diffusion. The resulting analytical Eulerian expression is shown to allow for a finite peak of the diffusion ratio for local Stokes numbers of order unity at eddy interaction times constant with the crossing-trajectory effect.

## Analysis

### One-Dimensional Eddy Description

We first assume that the particles do not significantly modify the average eddy size or strength, which is true for mass loadings much less than unity, e.g., less than 5% for Stokes numbers of order unity.[1,11] For the flow, we consider a two-dimensional reference frame for the shear layer (no spanwise variation) where the mean flow is in the $+x$ axis (streamwise) direction and, thus, transverse diffusion will be defined in the $\pm y$ direction. The two-dimensional velocity field is thus given by the component $u$ in the streamwise direction and $v$ in the transverse direction. Similar to the DRW Lagrangian stochastic models, the broad spectrum of turbulent eddies is simplified by local mean eddies that have uniform strength (based on the turbulence intensity), size (based on an integral scale), lifetime (based on the strength and size), and convection speed (equal to the mean streamwise convection velocity $u_m$). As is the case for the DRW models, this is based on ample experimental evidence that the large-scale structures primarily control turbulent dispersion for heavy particles.[2,12] As such, the current Eulerian model does not incorporate an energy spectrum as is included by the models of Reeks[8] and Stock.[9]

For turbulent free shear layers, the mean transverse fluctuations of the continuous fluid ($v'_{f,\mathrm{rms}}$) are proportional to the difference in velocity between the high-speed and low-speed streams ($u_1 - u_2 = \Delta u$) such that $v'_{f,\mathrm{rms}}$ is approximately $c_1\,\Delta u$, where is $c_1$ is a constant. The mean eddy size $\Lambda$ is approximated as an integral scale that is a fraction of the overall shear layer thickness ($\Lambda = c_2\delta$). Thus, the eddy lifetime is

$$\tau_e = c_2\delta / c_1\,\Delta u$$

Although Mei and Adrian,[13] as well as Hunt et al.,[10] note that one must be careful when expressing the Lagrangian integral timescale as a function of the integral scale and the rms of the velocity fluctuations, this form is reasonable for the qualitative model we are constructing.

The timescale of the fluid eddy interaction with the particle ($\tau_i$) must be described to determine the trajectory integration to obtain net particle deflection. The DRW model of Gosman and Ioannides[14] takes this timescale to be the minimum of the eddy lifetime and eddy traverse time ($\tau_{\mathrm{transit}}$, the time for the particle to cut through the eddy). We choose a similar but more smoothly varying function:

$$\frac{1}{\tau_i} = \frac{1}{\tau_e} + \frac{1}{\tau_{\mathrm{transit}}} = \left[\frac{c_1\Delta_u + V_{\mathrm{term}}}{c_2\delta}\right]$$

where we have assumed that the relative velocity of the particle with respect to the eddy ($V_{\mathrm{rel}}$) is approximately equal to the terminal velocity of the particle in quiescent flow ($V_{\mathrm{term}}$). Employing a crossing time based on the linearized equation of motion, as shown by Graham and James,[15] yields similar results. The preceding equation avoids the use of empirical constants to correct $\tau_e$ and $\tau_{\mathrm{transit}}$ to fit experimental data of particle diffusion. Also note that, if one examines dispersion in a theoretical steady Stuart vortex field,[16,17] the lifetime of the eddy is essentially assumed to be infinite, i.e., $\tau_i = \tau_{\mathrm{transit}}$.

Now, we define the one-dimensional temporal nature of the eddy by noting that its velocity perturbation will change during the interaction period such that the transverse fluid velocity $v_f$ seen by the particle is initially at zero, linearly increases until it peaks at $\tau_i/2$, and then linearly descends at the same rate until $3\tau_i/2$, i.e.,

$$\begin{aligned} v_f &= 2\sqrt{3}\varepsilon c_1 \Delta u t/\tau_i \qquad \text{for} \qquad t < \tau_i/2 \\ v_f &= 2\sqrt{3}\varepsilon c_1 \Delta u (1 - t/\tau_i) \qquad \text{for} \qquad t > \tau_i/2 \end{aligned} \tag{1}$$

The value $\varepsilon$ is included to consider both a transverse velocity perturbation, which initiates in the $+y$ direction ($\varepsilon = 1$), and a transverse velocity perturbation initiating in the $-y$ direction ($\varepsilon = -1$). The factor $2\sqrt{3}$ simply stems from specifying that $v_{f,\mathrm{rms}} = c_1\Delta u$ if integrated over any time period based on multiples of $\tau_i/2$. By basing this velocity on the interaction time (instead of the eddy lifetime), this velocity fluctuation is taken to be along the particle path (and not on the fluid path). The result is a description that is continuous in velocity but discontinuous (albeit finite) in acceleration, which is meant to qualitatively capture the temporal variations of the turbulent velocity (which rise and fall over finite time periods).

### Particle Equation of Motion

For monodisperse inert particles of constant mass and of much greater density than the surrounding fluid, the equation of motion relates the Lagrangian particle acceleration ($\mathrm{d}\boldsymbol{V}_p/\mathrm{d}t$) to drag, lift, and gravitational forces. The drag force $\boldsymbol{D}$ acts in the direction of the relative instantaneous velocity of the dispersed phase ($\boldsymbol{V}_{\text{rel}} = \boldsymbol{V}_f - \boldsymbol{V}_p$). We define a small particle as having a drag coefficient proportional to the inverse of the particle Reynolds number, i.e., $C_D = k_1/Re_p$, where $Re_p$ is the particle Reynolds number based on diameter $d$, relative velocity $V_{\text{rel}}$, continuous phase viscosity $\mu_f$, and density $\rho_f$, yielding $Re_p = \rho_f V_{\text{rel}} d/\mu_f$. For a solid Stokesian particle (with $Re_p$ of order unity), we note $k_1 = 24$. For a particle with a fluid interior, e.g., drops, and no surfactants, we find $k_1 = 16$ (Ref. 18). We can then define the drag force based on $\tau_p$:

$$\boldsymbol{D} = D_{\text{term}} \frac{\boldsymbol{V}_f - \boldsymbol{V}_p}{V_{\text{term}}} = m_p \frac{\boldsymbol{V}_f - \boldsymbol{V}_p}{\tau_p}$$

where the particle response time can be written as $\tau_p = V_{\text{term}}/g = (4\rho_p g d^2)/(3g\mu_f k_1)$ for a spherical particle, where $g$ is the acceleration due to gravity. Note that nonspherical particles can also be modeled if the drag coefficient is still inversely proportional to the Reynolds number (reasonable for $Re_p < 1$ for a large range of eccentricities[19]) and where the diameter is understood to mean the equivalent volumetric diameter. The gravitational force ($\boldsymbol{G} = m_p \boldsymbol{g}$) acts in the direction of gravity, which has streamwise and transverse components given by $g\theta_x$ and $g\theta_y$. The lift force (based on the mean spanwise vorticity) is dropped herein for conciseness from the rest of the analysis because its inclusion will not alter the resulting mean turbulent diffusion rates.[20] This is because the lift force is independent of $\varepsilon$ such that, even though it will cause a net drift of the mean particle trajectory, the modeled mean diffusion rate about this mean drift trajectory will not be affected. Forces associated with the fluid stress gradient can be neglected because the particle density is much greater than that of the surrounding fluid. In addition, the Basset history forces and those due to Faxen terms are herein neglected, which is reasonable for many, but not all, conditions.[21] As a result, our equation of motion then becomes

$$m_p \frac{\mathrm{d}\boldsymbol{V}_p}{\mathrm{d}t} = m_p \frac{\boldsymbol{V}_f - \boldsymbol{V}_p}{\tau_p} + m_p \boldsymbol{g}$$

To obtain the transverse deflections for our deterministic one-dimensional eddy, we must consider the $y$ component of the momentum equation:

$$\frac{\mathrm{d}v_p}{\mathrm{d}t} = \frac{v_f - v_p}{\tau_p} + g\theta_y$$

Recall that we are interested only in characterizing the mean transverse diffusion of the particles and not the instantaneous eddy-resolved spatial structures. As such we will ignore streamwise segregation effects and approximate the streamwise particle velocity as

$$u_p = V_{\text{term}}\theta_x + u_m$$

where the mean eddy convection speed is approximated simply as $u_m = (u_1 + u_2)/2$.

**Integration of Particle Motion**

The transverse component of particle acceleration will be integrated (from $t = 0$ to $\frac{3}{2}\tau_i$) to obtain transverse particle velocity and will be integrated again to obtain transverse particle position. This position compared with that for a scalar will allow the computation of the transverse particle diffusion ratio. We note that, for our simple model, total integration to times that are even-integer multiples of $\tau_i/2$ yields a zero net diffusion for the scalar. Therefore, we use the minimum odd-integer value that is capable of yielding nonlinear behavior of the particle trajectory.[20] This is reasonable because our objective is to understand the qualitative physics of the particle–eddy interaction. A more quantitative simulation could be obtained with longer integration times and a spectrum of eddy sizes and velocity fluctuations, as was done by Reeks[8] and Stock.[9]

The integration is conducted in two steps based on the composite description of transverse velocity given by Eq. (1). Consider the first portion of our temporal duration ($t < \tau_i/2$); for $\varepsilon = 1$ we have

$$\frac{\mathrm{d}v_p}{\mathrm{d}t} + \frac{v_p}{\tau_p} = 2\sqrt{3}c_1 \frac{\Delta u t}{\tau_i \tau_p} + g\theta_y \qquad (2)$$

To integrate, we assume that $v_p$ has the following form (for both the first and second parts of the temporal integration):

$$v_p = f_1 e^{(f_2 t)} + f_3 t + f_4$$

where $f_1$, $f_2$, $f_3$, and $f_4$ are constants. Substituting this form into the derivative of Eq. (2), terms of equal dependence on times are collected. The initial condition for $v_p$ at $t = 0$ (where $v_f = 0$) is obtained by assuming the particle is moving at terminal velocity prior to interacting with the eddy, i.e.,

$$(v_f - v_p)|_{t=0} = -V_{\text{term}}\theta_y = f_1 + f_4$$

Note that this initial condition can be modified without loss of generality if the local relative particle velocity is not reasonably represented by the terminal velocity. Integrating from this intermediate time to the final time with an acceleration of opposite sign [Eq. (1)] for $\varepsilon = 1$ yields the following velocity[20]:

$$v_{p,3\tau_i/2} = h_1 \exp(-3\tau_i/2\tau_p) - 3\sqrt{3}c_1\Delta u + 2\sqrt{3}c_1\Delta u(1 + \tau_p/\tau_i) + \tau_p g\theta_y$$

where

$$h_1 = [-4\sqrt{3}c_1\Delta u(\tau_p/\tau_i)]/\exp(-\tau_i/2\tau_p) + (V_{\text{term}}\theta_y - \tau_p g\theta_y + 2\sqrt{3}c_1\Delta u\tau_p/\tau_i)$$

Similarly, the net transverse displacement $y_{p,3\tau_i/2}$ is obtained by using the initial condition $y_p|_{t=0} = 0$ and integrating the general expression for particle velocity for both time durations of Eq. (1) to yield a closed-form (albeit lengthy) expression.[20] To reference the transverse deflection of the particle, the deflection of a continuous fluid marker (given by $\tau_p \to 0$) is subjected to the same velocity perturbation, which yields a simple expression:

$$y_{f,3\tau_i/2} = \sqrt{3}/4\varepsilon_0 c_1 \Delta u \tau_i$$

The ratio of particle deflection to scalar deflection for equal interaction times ($y_p/y_f|_{\tau_i}$) can then be determined. However, the deflection ratio for equal streamwise positions ($y_p/y_f|_{x_f}$) is more consistent with measurements of particle diffusion. To correct for this, it is assumed that the particle diffusion rate is approximately constant such that the transverse diffusion is linearly proportional to the mean streamwise convection for equal time, i.e.,

$$y_p|_{x_f} = y_p|_{\tau_i}(x_f/x_p)|_{\tau_i}$$

where the streamwise positions can be evaluated based on the following:

$$\left.\frac{x_f}{x_p}\right|_{\tau_i} = \frac{u_m}{u_m + V_{\text{term}}\theta_x}$$

Therefore, we have

$$\frac{y_p}{y_f|_{x_f}} = \frac{y_p/y_f|_{\tau_i}}{(1 + V_{\text{term}}\theta_x/u_m)}$$

**Particle Diffusion Ratio**

The ratio of local particle to scalar spread rates in a turbulent free shear layer can be used to give a physical interpretation of the mean diffusion ratio between the particle concentration and a scalar field concentration as follows:

$$\bar{D} = \left[\frac{1}{2}\frac{\mathrm{d}\overline{y_p^2(t)}}{\mathrm{d}t} \bigg/ \frac{1}{2}\frac{\mathrm{d}\overline{y_s^2(t)}}{\mathrm{d}t}\right] \approx \frac{b_p^2}{b_s^2}$$

where $\overline{y_p^2(t)}$ is the mean square particle displacement, $\overline{y_s^2(t)}$ is the mean square scalar displacement, $b_p^2$ is the square of the local spatial spread rate for the particles, and $b_s^2$ is the square of the local spatial spread rate for the scalar.

To concisely present the modeled particle diffusion ratio, four dimensionless parameters ($S$, $\zeta$, $\beta$, and $Fr_\delta$) are introduced. The first parameter is the ratio of the particle and eddy timescales ($\tau_p/\tau_e$) defined herein as the local Stokes number

$$S = \frac{\tau_p}{\tau_e} = \frac{4}{3}\frac{d^2\rho_p c_1 \Delta u}{k_1 \mu_f c_2 \delta}$$

Note that $S$ is equal to the conventional Stokes number[2] under the following constraints: $c_1 = 1$, $c_2 = 1$, and $k_1 = 24$. The second parameter is a drift parameter,[9] which is the ratio between particle terminal velocity and fluid turbulence intensity:

$$\gamma = \frac{V_{\text{term}}}{v'_{f,\text{rms}}} = \frac{V_{\text{term}}}{c_1 \Delta u}$$

Note that two limits of the particle description can be defined based on this parameter: a slow particle, which has a terminal velocity smaller than that of the characteristic transverse turbulent velocity of the surrounding fluid ($\gamma < 1$), and a fast particle, where the opposite is true ($\gamma > 1$). The third parameter is the ratio between interaction time and eddy timescale, i.e., $\beta = \tau_i/\tau_e$. The fourth parameter is the eddy Froude number,[22] defined as $Fr_\delta = \Delta u^2/4g\delta$, which is independent of the particle characteristics.

The particle diffusion can be related to the difference between the two transverse particle positions given by $\varepsilon = 1$ ($v_f$ initially positive) and $\varepsilon = -1$ ($v_f$ initially negative) normalized by the streamwise movement: $(y_p|_{x_f,\varepsilon=1} - y_p|_{x_f,\varepsilon=-1})/x_f$. This can be compared to that for a fluid tracer to obtain the particle deflection ratio $\phi$

$$\phi = \left(y_p|_{x_f,\varepsilon=1} - y_p|_{x_f,\varepsilon=-1}\right)\big/\left(y_f|_{x_f,\varepsilon=1} - y_f|_{x_f,\varepsilon=-1}\right)$$

The portion of the deflections due to drift from gravity and lift in the transverse direction will not contribute to $\phi$ because of symmetry (they are independent of $\varepsilon$ and will cancel out). The particle deflection ratio is assumed to equal the local ratio of particle spread rate to a fluid tracer spread rate:

$$\bar{D} = \frac{\left[1 + 4S/\beta - 8S^2/\beta^2 - \{(8S^2/\beta^2)e^{(-\beta/2S)} - 16S^2/\beta^2\}e^{(-\beta/S)}\right]^2}{(1 + c_1\theta_x \Delta u \gamma/u_m)^2} \tag{3}$$

where we summarize:

$$S = \frac{4}{3}\frac{d^2\rho_p c_1 \Delta u}{k_1 \mu_f c_2 \delta}, \qquad \beta = \frac{1}{1+\gamma}$$
$$\gamma = \left[\frac{S}{Fr_\delta}\right]\left[\frac{c_2}{4c_1^2}\right], \qquad Fr_\delta = \frac{\Delta u^2}{4g\delta} \tag{4}$$

Note that the ratio $S/\beta$ that appears in Eq. (3) is simply the ratio of particle response time to the interaction time, i.e., $\tau_p/\tau_i$. Equations (3) and (4) thus provide a closed-form expression for the local mean diffusion ratio of a small particle. The result indicates three important nondimensional parameters: local Stokes number $S$, eddy Froude number $Fr_\delta$, and drift parameter $\gamma$, of which two are independent parameters, e.g., $S$ and $Fr_\delta$.

Although previous Eulerian models (as described earlier) typically include dependence on only one of these parameters, the preceding conclusion regarding two independent controlling parameters is consistent with previous analytical studies that can be related to the understanding of $\bar{D}$. For example, the independent parameters $S$ and $\gamma$ would be consistent with two of the parameters identified in the analytical studies of Stock,[9] as well as Mei and Adrian,[13] for the determination of particle dispersion in a high Reynolds number turbulent spectrum. Similarly, Hunt et al.[10] use heuristic arguments to suggest these same two independent parameters. Gañán-Calvo and Lasheras[16] also noted that the parameters $S$ and $S\gamma$ are critical to particle trapping in Stuart vortices. Finally, the analytical study by Reeks[8] investigated the diffusion as a function of parameters similar to $S$ and $Fr_\delta$.

## Results

### Parametric Influence

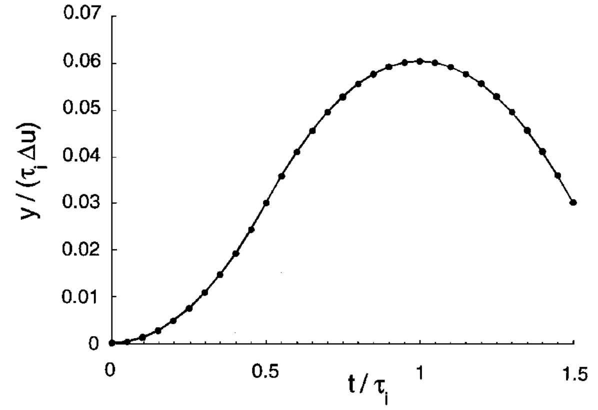
For the predictions, only two of the three fundamental diffusion parameters ($S$, $Fr_\delta$, and $\gamma$) can be considered independent. We have chosen herein to consider variations based on $S$ because it has the strongest influence and on $Fr_\delta$ because it is independent of particle properties. To predict the diffusion in a free shear layer, we also need to approximate $c_1$ and $c_2$. This is accomplished by employing experimental data for incompressible constant-density planar turbulent free shear layers (for which there are quantitative experimental data). We may estimate $c_1$ by assuming that the average value across the shear layer is equal to one-half the measured peak value of $v'_{f,\text{rms}}/\Delta u$. Based on results given by Wygnanski and Fiedler,[23] $c_1$ is approximately 0.07. For $c_2$, we can estimate the integral scale as a fraction of the shear layer thickness based on velocity $\delta$. Based on results discussed by Oakley et al.,[24] $c_2$ is approximately 0.33. We will, therefore, use these values and $k_1 = 24$ for all of the predictions.

Figure 1 shows plots of nondimensional particle and fluid tracer transverse deflections ($y/\tau_i \Delta u$) as a function of time for the proposed one-dimensional eddy for $Fr_\delta = 100$. For infinitely small particles ($S = 0$), there is no difference between the particle trajectory and that of the fluid tracer, i.e., the particle reacts instantly to the transverse velocity fluctuations. For $S = 0.1$, we find that the particles lag behind the fluid transverse fluctuations and thus tend to overshoot the fluid motion at $t = \frac{3}{2}\tau_i$. This is consistent with the experimentally observed nonmonotonic behavior, where particles are dispersed outward but then cannot respond quickly enough to return with a subsequent opposing velocity perturbation. For $S = 1$, the net deflection for both is about the same (although the strong differences in the time evolution of the trajectories suggest that this condition would yield significant variations in the spatially resolved particle concentration as compared to that of the scalar concentration, i.e., they may have similar mean diffusion but different dispersion structures). For $S = 10$, the particle can respond only weakly to the transverse fluctuations due to its increased inertia, and the result is a tendency for the particle to remain near the $x$ axis.

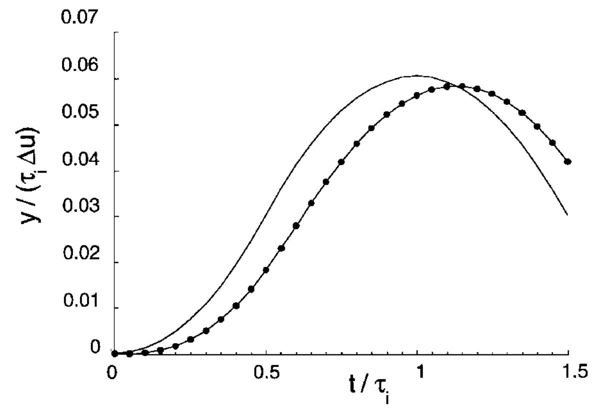
Figure 2 shows the change in $\bar{D}$ as $S$ and $Fr_\delta$ vary. For fixed conditions of the continuous phase ($Fr_\delta$ const): $\bar{D} \to 1$ as $S \to 0$ (particle diffusion is identical to scalar diffusion); $\bar{D} > 1$ as $S \sim 1$ (particle diffusion is greater than scalar diffusion); and $\bar{D} \to 0$ as $S \to \infty$ (no particle diffusion). These results are consistent with those seen experimentally, e.g. Refs. 2 and 25–27 (note that the diffusion results for Ref. 27 were obtained in Ref. 20). The resulting influence of $Fr_\delta$ yields the family of curves relating local Stokes number effects on diffusion, as noted in Fig. 2. We note an increase in the $S$ corresponding to the peak location of $\bar{D}$ as $Fr_\delta$ increases; this is a result of the particle–eddy interaction being more controlled by the eddy lifetime than the eddy transit time. In general, diffusivity for larger particles increases with increasing eddy Froude number for a fixed Stokes number. This is consistent with the analytical results of Stock,[9] which showed that, for constant Stokes number, as the drift parameter decreases [consistent with eddy Froude number increasing as per Eq. (4)] the diffusivity increases and is attributed to the crossing-trajectory effect, a result that has also been confirmed in experimental studies. Note also the large differences that can occur at a fixed $S$ value for the predicted $\bar{D}$ as $Fr_\delta$ changes, especially at low values of $Fr_\delta$. Consequently, the eddy Froude number is an important diffusion parameter for fast particles ($V_{\text{term}} > v'_{f,\text{rms}}$) but not for slow particles ($V_{\text{term}} < v'_{f,\text{rms}}$).

### Experimental Comparisons

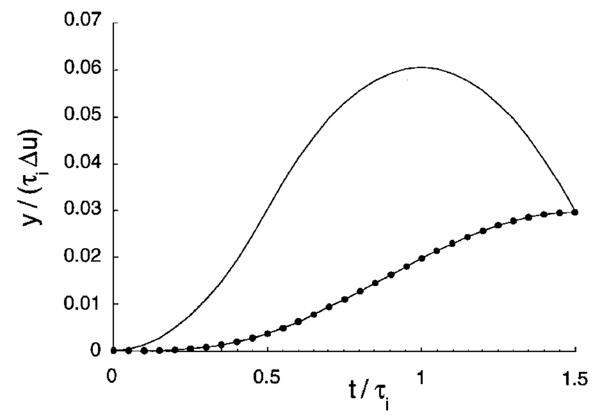
To investigate the ability to qualitatively represent particle diffusion for turbulent free shear layers, the model is evaluated with the experimental data. Unfortunately, mean diffusion ratios in free shear flows are notoriously hard to quantify. Considerable uncertainty and/or bias was found for many of the data sets cited in the literature that specifically reported $\bar{D}$ greater than unity, rendering their relationship between diffusion ratio and local Stokes number values to be primarily qualitative.[7] The data set of Hishida et al.[25] was chosen because it was deemed to contain highest-quality measured values of the diffusion ratio while still reporting diffusion ratios greater than unity. However, even the data of Hishida et al. have significant uncertainty, e.g., it is estimated that the values of



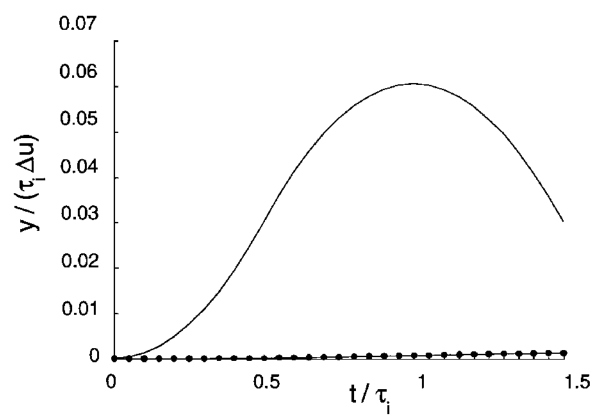

a) $S = 0$




b) $S = 0.1$




c) $S = 1$




d) $S = 10$

**Fig. 1 Particle trajectories ( · · · · · ) in time as compared to fluid tracer trajectories (——) for $Fr_\delta = 100$.**

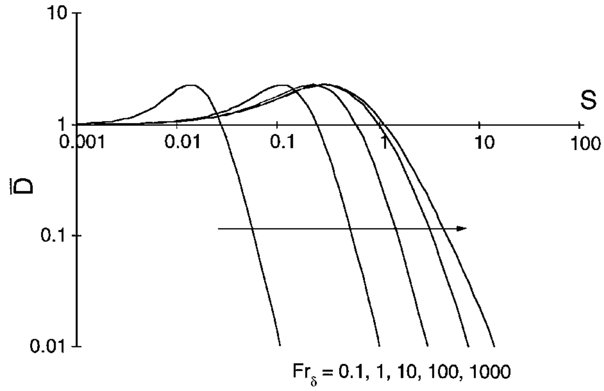



**Fig. 2 Particle diffusion ratio vs local Stokes number at various eddy Froude numbers.**

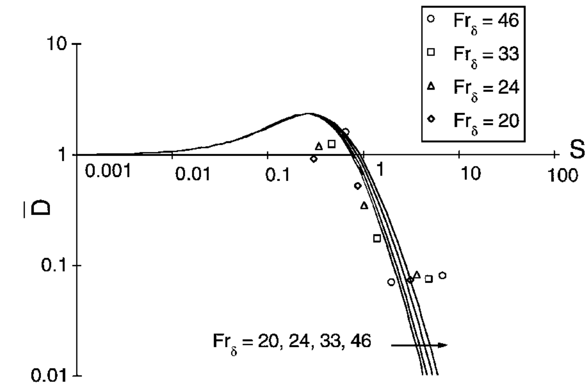



**Fig. 3 Experimental diffusion ratios of Hishida et al.[25] as compared to model predictions based on estimated local Stokes number and eddy Froude number.**

both $S$ and $\bar{D}$ may be off by as much as a factor of two based on their reported methodology and the present definitions. For estimating $S$ from Hishida et al., $\delta$ was based on their streamwise width of the coherent structure, which was calculated based on the peak frequency of their spectrum and the convection speed. Three different particle sizes and four different streamwise locations yielded a total of 12 different $S$ values. The corresponding diffusion ratio $\bar{D}$ was calculated by dividing the ratio of particle diffusion to momentum diffusion by an estimate of 0.7 for the turbulent Schmidt number (ratio of momentum to diffusion and scalar diffusion).

Figure 3 shows the experimental data and the curves of the proposed model at the respective experimental Froude numbers. Both show the same type of nonmonotonic $\bar{D}$ trend noted in previous studies.[1] Although the experiments show a significantly narrower region of $S$ values that yield $\bar{D} > 1$, the predictions reveal qualitative agreement with respect to the influence of both $S$ and $Fr_\delta$, i.e., increasing the eddy Froude number tends to increase the diffusion for equivalent local Stokes numbers. Thus, the present model is reasonably descriptive of the qualitative particle–eddy interactions despite its simplicity and nonempiricism. However, given the small range of eddy Froude number and the experimental uncertainty for this data set, this trend is not conclusively confirmed. Note that the purpose was not to predict a specific experimental data set (employing empirical coefficients of order unity would have been used if that had been the case) but to determine whether a qualitative Eulerian mean diffusion model could be constructed using concepts developed in the Lagrangian diffusion approaches.

The practical importance of this model may lie in its eventual use in Eulerian RANS computations of mean particle concentration fields, although more work is certainly needed to make the model quantitative. A more accurate model of the turbulent diffusion could be obtained by incorporating a variety of initial conditions, longer integration times, and perhaps most importantly a spectrum of both eddy sizes and velocity perturbation strengths. In addition, comparison with other high-accuracy diffusion data at different test conditions would be beneficial.

## Conclusions

A closed-form analytic expression for the local mean diffusion of a particle (or droplet) was developed for a turbulent free shear layer when the drag coefficient is inversely proportional to particle Reynolds number. The model assumes a one-dimensional eddy that includes finite fluid acceleration and is based on a local length and timescales of the turbulence. The resulting diffusion indicates three important nondimensional parameters: a local Stokes number $S$ (ratio of particle response time to local eddy lifetime), the eddy Froude number $Fr_\delta$ (ratio of rotational acceleration to gravitational acceleration), and the drift parameter $\gamma$ (ratio of particle terminal velocity to local turbulent rms fluctuations); of these three parameters, two are independent. The model predicts the correct qualitative trends observed in experimental particle diffusion ratios of planar turbulent free shear layers, i.e., a peak particle diffusion ratio occurring at near unity local Stokes number followed by monotonic reduction in the diffusion as the Stokes number is increased. The shift in this peak with eddy Froude number found in the model was not conclusively verified with the experiments. Additional work is required to encompass the model in a RANS-type framework for general free shear flows and for comparison with high-accuracy experimental data.

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